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1992 J. Phys. A: Math. Gen. 25 5823

(http://iopscience.iop.org/0305-4470/25/22/010)

J. Phys. A: Math. Gen. 25 (1992) 5823–5830. Printed in the UK

Anisotropy in field-assisted diffusion on a gasket fractal

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Received 12 March 1992

Abstract. The random walk on the two-dimensional gasket fractal has been studied with various forms of constant bias fields applied using the Monte Carlo method. Crossover from anomalous diffusion to drift was observed in the horizontal and upward bias fields while crossover from anomalous to normal diffusion was found in the downward bias field. The scaling relation $R(B_s, t) \sim t^k f(B_s t^k)$ was confirmed and the scaling exponent of $f(x) \sim x^{\beta}$ at large x was obtained as $\beta = 1.175 \pm 0.004$ for the horizontal bias field and $\beta = 0.319 \pm 0.002$ for the downward bias field in good agreement with the theoretical work by Parrondo *et al.*

1. Introduction

Diffusion under a constant external bias field has been studied on random fractals; for example, self-avoiding walks (Chowdhury 1985) and percolation clusters at the percolation threshold (Barma and Dhar 1983, Pandey 1984, Dhar 1984, Stauffer 1985). In a uniform structure the effect of the bias field is to produce a drift in the direction of the applied field. In a random structure the constant bias field has two competing effects: the bias field induces drift in the direction of the field but also creates traps such as dangling ends and backbending in the percolation clusters (Dhar 1984, Ohtsuki and Keyes 1984, Stauffer 1985). The root mean square (RMS) displacement of the random walker under the constant bias field in a random structure behaves as $R(B_s, t) \sim t^k$ with $k \leqslant 1$. At short times diffusion ($k \leqslant 1/2$) dominates and at long times drift ($k = 1$) dominates. The crossover from diffusion to drift behaviour occurs at the crossover time $t^* \sim 1/B_s$ below the characteristic bias field B_s^*. Above B_s^* the trapping induced by the applied field leads to a logarithmic slow increase in the RMS displacement $R(t) \sim (\ln(t))^{\gamma}$ of the random walk with $\gamma = 1$ in the percolation cluster (Havlin and Ben-Avraham 1987).

A random walk on a two-dimensional Sierpinski gasket with a bias field has been studied theoretically using scaling arguments by Stinchcombe (1985) and numerically using the Monte Carlo method by Kim *et al* (1987). They proved the existence of a bias-field-induced crossover from anomalous diffusion ($k = 0.431$) to drift ($k = 1$) and the crossover time t^* was a decreasing function of the bias field. However, there was no logarithmic slowdown in the strong field due to the absence of the strong trapping geometry such as the dangling ends of the percolation cluster and no characteristic bias field B_s^*.

Recently Parrondo *et al* (1990) have studied a biased random walk on a two-dimensional gasket using renormalization procedures for the hopping probabilities and waiting time. They obtained a different behaviour for the anisotropic diffusion for the downward applied field of figure 1 and proposed a scaling relation which described the crossover from anomalous diffusion to drift in the horizontal field and the crossover from anomalous diffusion to normal diffusion ($k = 1/2$) in the downward field. The scaling exponent β at long times was predicted to be $\beta = (1-k)/k$ for the horizontal field and $\beta = (1-2k)/2k$ for the downward field. In the present work we want to confirm the scaling relation and the directional dependence of the random walk to obtain the scaling exponents using Monte Carlo simulation.

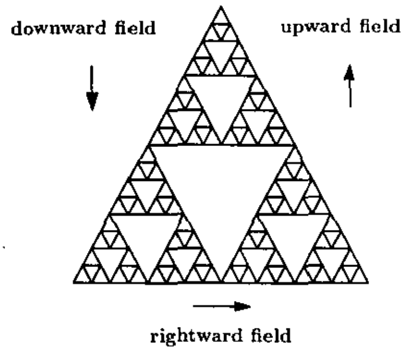

Figure 1. Sierpinski Gasket ($n = 3$) and the directions of the constant bias field.

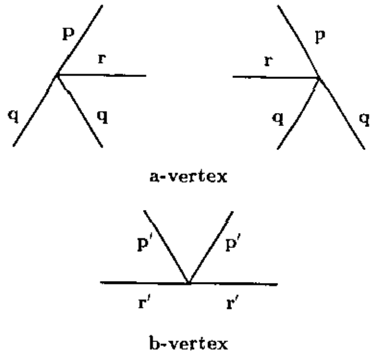

Figure 2. The two types of vertex on the Sierpinski gasket under the constant bias field are associated with different hopping probabilities.

2. Theory

The random walk in fractals is described by the anomalous diffusion exponent $k < 1/2$. With no bias field

$$R(t) \sim t^k \tag{1}$$

where $R(t)$ is the RMS displacement from the local origin of the random walker and t is the time in units of the lattice constant and inverse jump rates (Bouchaud and Georges 1990a). In the presence of the constant bias field the hopping probability depends on the direction of the bias field and the vertex types as in figure 2. In the short-time regime the response of the random walker is linear in the applied field (Bouchaud and Georges 1990a,b)

$$R(B_s, t) \sim B_s t^k \tag{2}$$

where $R(B_s, t) B_s < k_B T$, i.e. random (or thermal) hopping probabilities are greater than the bias hopping, and the linear response always holds. In the long-time regime there is no general expression for the RMS displacement $R(B_s, t)$. In disordered lattices where the diffusion is normal one obtains an Einstein relation for a weak field (Bouchaud and Georges 1990a)

$$R(B_s, t) \sim \frac{D}{k_B T} B_s t \tag{3}$$

where D is diffusion constant and k_B is Boltzmann's constant. When the trapping of the structure under the bias field is not strong, we can apply the Pincus blob picture of a stretched polymer made of n-monomers (Pincus 1976, De Gennes 1985, Bouchaud and Georges 1990a). For a length scale less than ξ_B the walk is considered in the same way as in the zero field case. For long times the energy gain due to the field is greater than k_BT. Therefore, for the length scale larger than ξ_B, the field dominates and the walk drifts along the direction of the field. The length scale ξ_B ($B\xi_B = k_BT$) is the crossover length between the zero field ($B_s = 0$) and infinite field ($B_s = 1$) regimes. In the long-time regime

$$R(B_s, t) \sim \frac{t}{t^*}\xi_B \tag{4}$$

where t^* is the mean time needed to cross the length ξ_B. If t^* is finite, one obtains $\xi_B \sim t^{*k}$. By using $B\xi_B = k_BT$ one obtains the crossover time as

$$t^* \sim (k_BT/B_s)^{1/k}. \tag{5}$$

This crossover time t^* is seen to decrease with increasing field and the anomalous diffusion exponent k gives the structural characteristics for crossover. The RMS displacement of the random walker at $t > t^*$ becomes

$$R(B_s, t) \sim t\xi_B^{1-1/k} \sim tB_s^{(1-k)/k}. \tag{6}$$

The response is nonlinear in field B_s and shows drift behaviour at $k = 1$. Equation (6) may be applied to the horizontal and upward bias fields in the two-dimensional gasket fractal as the structure conforms to that of a stretched polymer. For downward bias of the gasket the response is different from that of the horizontal bias field in the long-time limit. Parrondo *et al* obtained the dependence of the RMS displacement for the downward bias field at long times. They found the renormalized hopping probabilities using the decimation method for the two types of vertex in figure 2 and the fixed points were $q = r' = 1/2$ and $r = p = p' = 0$. These fixed points imply that the hopping probabilities in the direction of the bias field increase with increasing intensity of the bias field. That is, the probability (q) of escaping the a-vertex increases in the direction of the bias field and decreases in the other directions. At the b-vertex escape is less likely in the direction opposing the field and thus more likely along the directions perpendicular to the field. For a high field the waiting time at the a-vertex is negligible due to the strong hopping probability in the q direction. But the remaining time at the b-vertex at each step of the walk is very large because of the symmetric hopping probability in the r' direction. Hence the walk in the horizontal direction that the walker must perform before it moves from a b-vertex to an a-vertex corresponds to a one-dimensional random walk and one expects the mean time of escape to increase as a square of the length of the path. The RMS displacement derived by Parrondo *et al* (1990) was

$$R(B_s, t) \sim t^{1/2}B_s^{(1-2k)/2k}. \tag{7}$$

The response to a downward field has a different field dependence compared with that for the horizontal and upward fields. Normal diffusion with $k = 1/2$ arises

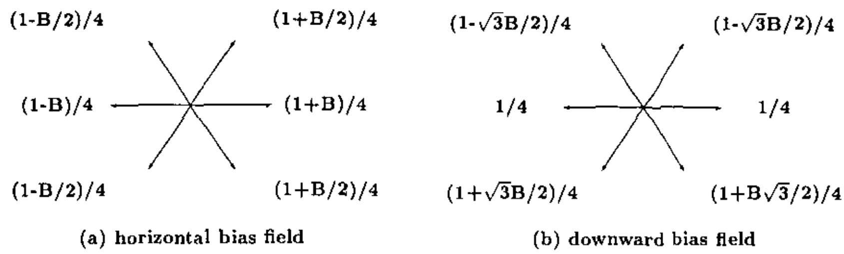

Figure 3. Anisotropic transition probabilities encountered in the Sierpinski gasket under the constant bias field: (*a*) rightward bias field and (*b*) downward bias field.

under fields at long time. The general response in the bias field can be described by a scaling form

$$R(B_s, t) \sim t^k f(B_s t^k) \tag{8}$$

where $f(0) = 1$ and $f(x) = x^\beta$ for large x. By the self-consistency check of the scaling relation of $R(B_s, t)$ at long times the scaling exponent β can be determined (Parrondo *et al* 1990).

In the horizontal bias field,

$$\beta = \frac{1-k}{k} = 1.320. \tag{9}$$

In the downward bias field,

$$\beta = \frac{1-2k}{2k} = 0.160. \tag{10}$$

The scaling exponent β determines the field dependence of the RMS displacement at long times.

3. Monte Carlo method

We generated the two-dimensional Sierpinski gasket up to ten stages ($n = 10$). The total number of sites in a d-dimensional Sierpinski gasket fractal is given (Hilfer and Blumen 1984) by $N_n = (1 + d)(1 + (1 + d)^n)/2$. In the two-dimensional gasket of ten stages the total number of sites is 88 575. The Monte Carlo method is employed to calculate the RMS displacement $R(B_s, t)$ of the random walker. In the isotropic case, i.e. unbiased case, the hopping probabilities are equal for the four nearest neighbours at any vertex. In the anisotropic case, i.e. with the constant bias field, the hopping probabilities depend on the direction of the bias field. Figure 3 shows the normalized hopping probabilities for each of six possible directions at a vertex. For the horizontal bias field in figure 3(*a*) the rightward hopping probabilities are enhanced by the bias field and for the downward bias field in figure 3(*b*) the downward hopping probabilities are enhanced. The intensity of the bias field was varied between zero ($B_s = 0$) and infinity ($B_s = 1$). Given a directional bias field B_s, we have randomly chosen a starting site for a random walker (blind ants) and have performed the random walk by generating a sequence of random numbers. The RMS displacement $R(B_s, t)$ of the random walker for a given B_s was averaged over 5000 starting local origins. The random walk was performed for the three directions of the bias field shown in figure 1 and various intensities of the bias field.

4. Results and discussions

With no bias field we could reproduce the anomalous diffusion of the random walker represented by the lowest line in figure 4, from which we could obtain the anomalous diffusion exponent $k = 0.432$. In figure 4 we show the calculated RMS displacement $R(B_s, t)$ under the various rightward bias fields. The crossover from anomalous to drift ($k = 1$) is consistent with previous results (Stinchcombe 1985, Kim *et al* 1987). The crossover time t^* is seen to decrease with increasing field, which is in agreement with the prediction of equation (5). But there is no characteristic field B^* above which there is logarithmic diffusion $R(B_s > B^*, t) \sim \ln(t)$ of the percolation cluster. The absence of the characteristic field B^* means that there is no trapping geometry such as dangling ends or backbends (or cages). For the strong bias field the boundary effect appears as a flat plateau, but it does not prevent the observation of crossover behaviour. Our simulation employs reflection at the boundaries. In both the horizontal and downward strong fields particles reach the boundaries too soon to observe the proper long-time behaviour. When the field was reversed to the leftward direction, we observed the same response as for the rightward direction.

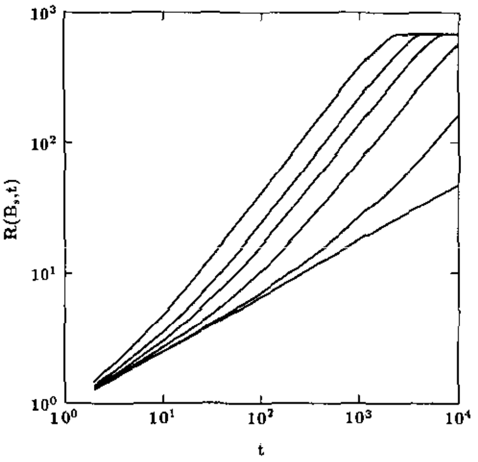

Figure 4. Log–log plot of the RMS displacement $R(B_s, t)$ against time under the various rightward constant bias fields of $B_s = 0.0$, 0.1, 0.5, 0.7, 1.0 from below.

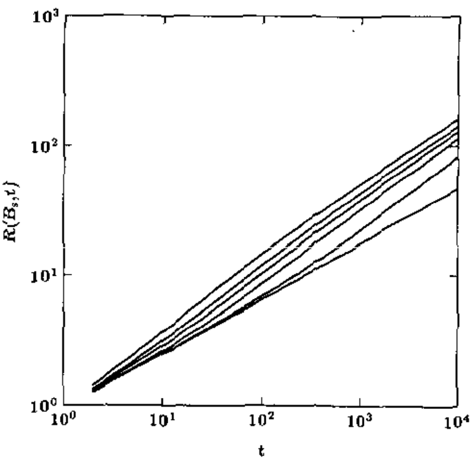

Figure 5. Log–log plot of the RMS displacement $R(B_s, t)$ against time under the various downward constant bias fields of $B_s = 0.0$, 0.1, 0.3, 0.5, 0.7, 1.0 from below.

Figure 5 shows the RMS displacement $R(B_s, t)$ for various downward bias fields. The responses of the random walker are much slower than those for the horizontal bias field. The crossover from anomalous diffusion ($k = 0.431$) to normal diffusion ($k = 0.5$) at long times and the decreasing crossover time t^* with increasing bias field, consistent with equation (5), are all observed. These numerical observations seem to support the renormalization group results of Parrondo *et al* (1990). In the downward bias fields the hopping probability (q) at the a-vertex and the probability (r') at the b-vertex are enhanced strongly by the bias field and the opposing hopping probabilities such as p of the a-vertex and p' at the b-vertex are suppressed rapidly. Thus the remaining time at the a-vertex is very short and the motions of the random walkers are determined by r' of the b-vertex. Because the left and right hopping probabilities r' are same, the random walkers follow quasi one-dimensional diffusion in the direction perpendicular to the applied bias field. Therefore we can observe normal diffusion, $R(B_s, t) \sim t^{1/2}$, in the case of the downward bias field.

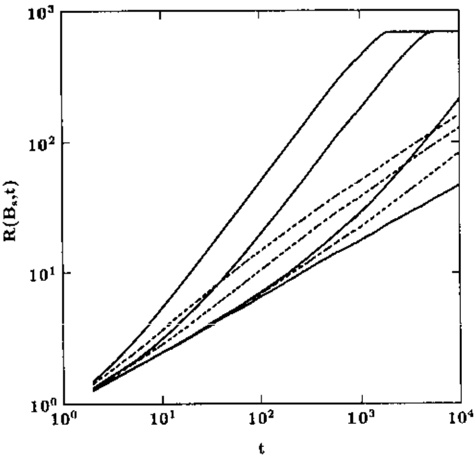

Figure 6. Log–log plot of the RMS displacement $R(B_s, t)$ against time under the various vertical constant bias fields of $B_s = 0.1$, 0.5, 1.0 from below. The full curves are for upward fields and the dotted curves are for downward fields. The lowest one corresponds to no bias field.

Figure 6 shows the response for the upward (full curves) and downward (dotted curves) bias fields of varying intensity. The RMS displacement $R(B_s, t)$ for the upward bias field is similar to the response in the horizontal bias field. We observed drift behaviour at long times and the crossover time t^* also decreases with increasing field. A similar random walk between a horizontal field and upward field leads to a similar structural growth for the gasket fractal with the corresponding bias field. The different results for the two reversed vertical fields means that there is strong anisotropy induced by the bias field for the vertical direction in the two-dimensional gasket fractal.

Figure 7 shows log–log plots of $R(B_s, t)/t^k$ against $B_s t^k$ to test the scaling relation of equation (8) for rightward bias fields with $k = 0.431$. At short times the collapse of data for various fields confirmed the scaling function as $f(0) = 1$ below crossover time t^*. At long times the data increase linearly in the log–log plot, which proves the scaling relation of $f(x) = x^\beta$ where $x = B_s t^k$. By least-squares fitting in the linear regime we find the scaling exponent β to be equal to 1.175 ± 0.004 which is smaller than the prediction of equation (9). At longer times the data bend down, which corresponds to the flat plateau in figure 4 and is due to the boundary effects. The scaling exponent β shows that the field dependence of the RMS displacement $R(B_s, t)$ is compatible with equation (6) applied for a horizontal field. The present result of small β compared with that of equation (9) indicates less dependence on the applied field by the RMS displacement.

In figure 8 we attempt to show the scaling relation by a log–log plot of $R(B_s, t)/t^k$ against $B_s t^k$ for the downward bias field. Compared with the horizontal field case in figure 7 the collapse of data is not so good for long times. In particular a larger deviation is seen in the results for the infinite field ($B_s = 1$) than in the data for lower fields. However, the scaling relation is observed to be $f(0) = 1$ at short times and $f(x) = x^\beta$ at intermediate times. The scaling exponent $\beta = 0.319 \pm 0.02$ could be derived for intermediate times. This value of β is twice the prediction $\beta = 0.160$ of equation (10). The origin of this higher value for the scaling exponent

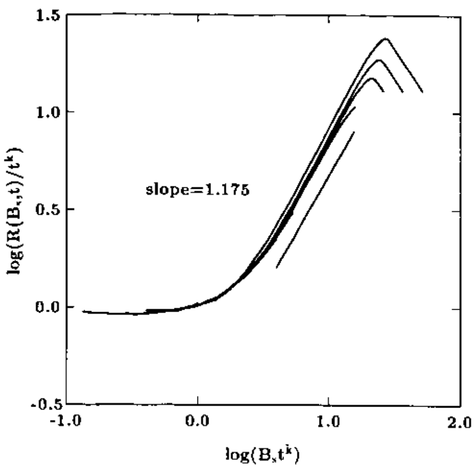

Figure 7. The scaling relation $R(B_s, t) \sim t^k f(B_s t^k)$ for the various rightward bias fields of $B_s = 0.1, 0.3, 0.5, 0.7, 1.0$ from below. The scaling exponent is obtained as $\beta = 1.175 \pm 0.004$.

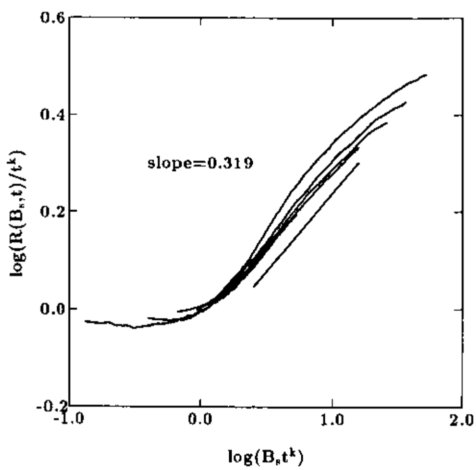

Figure 8. The scaling relation $R(B_s, t) \sim t^k f(B_s t^k)$ for the various downward bias fields of $B_s = 0.1, 0.3, 0.5, 0.7, 1.0$ from below. The scaling exponent is obtained as $\beta = 0.319 \pm 0.002$.

β may be that the waiting times at the a-vertex are not negligible compared with the waiting times at the b-vertex in field. The hopping probabilities r and q at the a-vertex and p' at the b-vertex in the field still have strong effects on the random walk. At longer times the slope decreases slowly, which is not the boundary effect of figure 7. In the strong downward field particles starting from the top sites are found to take longer to reach the boundary than those of random origins. At $B_s = 1$ the slope of the scaling relation curve for the top starting particle tends to decrease at long times compared with that for the random starting particle, which implies a decreasing value for the exponent β closer to the theoretical value of Parrondo *et al* (1990). We speculate that a possible transition to $\beta = 0.160$ at very long times will be observed in a much larger gasket fractal than our present system.

5. Conclusion

The bias field dependence of the random walk shows varying responses for different directions of the bias field. In the horizontal and upward fields crossover from anomalous diffusion to drift was observed but crossover from anomalous to normal diffusion was seen in the downward field. The scaling relation in the horizontal field was well observed at both short and long times but in the downward field the scaling relation at long times was not so convincing. The time dependence of the biased random walk at long times was in good agreement with the theories of Parrondo *et al* (1990) but the field dependence was not in full agreement with the results of the same theory. To observe the scaling exponent β properly in a strong downward bias field in the region of $x \gg 1$ it seems to be necessary to let the particles start at the top in a larger gasket of $n > 10$ and increase the Monte Carlo steps t.

Acknowledgments

This work was supported in part by the Korea Science and Engineering Foundation (Centre for Thermal and Statistical Physics).

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